

Relativistic electrons in a rotating spherical magnetic dipole: localized three-dimensional states

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Paralleling a previous paper, we examine single- and many-body states of relativistic electrons in an intense, rotating magnetic dipole field. Single-body orbitals are derived semiclassically and then applied to the many-body case via the Thomas-Fermi approximation. The many-body case is reminiscent of the quantum Hall state. Electrons in a realistic neutron star crust are considered with both fixed density profiles and constant Fermi energy. In the first case, applicable to young neutron star crusts, the varying magnetic field and relativistic Coriolis correction lead to a varying Fermi energy and macroscopic currents. In the second, relevant to older crusts, the electron density is redistributed by the magnetic field.

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1. INTRODUCTION

In a previous paper [1] (hereafter paper I), we examined the relativistic semiclassical relativistic orbitals of a particle of mass m and charge q in an intense magnetic field, idealized as a dipole, in a rotating reference frame. The particle was confined to the spherical surface. In this paper, we present the treatment of three-dimensional orbitals, the local cyclotron or Landau states. These results are applied to many-body electron states defined in a semiclassical (Thomas-Fermi) approximation and calculated in a simplified neutron star crust model with electrons and nuclei. We consider quantum dynamics only for the electrons, not the hadrons [2], but effects of relativity and rotation are included.

We ignore gravity (derivatives of the metric) here, as this is negligible for charged particles compared to the magnetic field. Where numerical values are needed, the tilt angle θ_0 between the dipole and rotation axes is assumed to be maximal, $\sin\theta_0 = 1$, and the rotational velocity Ω to be $\bar{\omega} = \Omega R/c = 0.01$, a realistic value for high-field neutron stars with radius R [3,4].

The treatment is based on expanding the particle motion in inverse powers of the field strength. Although electrons are stripped from the neutron star surface by

the rotation-induced electric field, the bulk of electrons remain in the crust to preserve local charge neutrality, with the surface sheathed by a thin space charge. The space charge is stabilized by the Coulomb force (with the positive crystal) opposing the induced electric field. Only a small fraction of electrons are accelerated into the stellar wind [5]. Our final applications here are to quantum single- and many-body states where radiation emission is neglected. This is exact for charged particles in their ground states or in excited states unable to decay by Pauli exclusion blocking in the presence of other fermions. We also seek a general classification of possible orbitals based on the relevant kinematic parameters. The semiclassical quantization is based on the Wilson-Sommerfeld or Bohr-Sommerfeld rule, a result of the WKB approximation [6].

FIG. 1. Geometry of the magnetic dipole \mathbf{M} and sphere rotating at angular velocity Ω , with relative tilt θ_0 .

2. GENERAL RELATIONS

2.1 Coordinates, metric, and field

As the magnetic field dominates rotational effects (unless r is quite large), we place the rotation axis at an angle θ_0 with respect to the magnetic dipole in the $\phi = 0, \pi$ plane (Fig. 1). The metric in a spherical polar coordinate system (r, θ, ϕ) rotating with the sphere is given by the line element

$$\begin{aligned}
ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\
&= c^2(1 - \boldsymbol{\omega}^2) dt^2 - dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) - \\
&\quad 2cr\omega_\phi \sin \theta dt d\phi - 2cr\omega_\theta dt d\theta \quad . \quad (1)
\end{aligned}$$

The vector $\boldsymbol{\omega}$ is defined from the rotational angular velocity vector $\boldsymbol{\Omega}$ by $\boldsymbol{\omega} = \boldsymbol{\Omega} \times \mathbf{r}/c$. We also use $\bar{\omega} \equiv \Omega R/c$, where $r = R$ is a reference sphere. The components of $\boldsymbol{\omega}$ are

$$\begin{aligned}
\omega_\phi &= (\bar{\omega}r/R)[\cos \theta_0 \sin \theta - \sin \theta_0 \cos \theta \cos \phi] \quad , \\
\omega_\theta &= -(\bar{\omega}r/R) \sin \theta_0 \sin \phi \quad , \\
\boldsymbol{\omega}^2 &= \omega_\phi^2 + \omega_\theta^2 \quad . \quad (2)
\end{aligned}$$

These are the appropriate generalizations of paper I and the treatment of Landau & Lifshitz [7] to the case $\sin \theta_0 \neq 0$ and $r \neq \text{constant}$. The lightsphere is the surface $\boldsymbol{\omega}^2 = 1$, or

$$\begin{aligned}
&\bar{\omega}^2(r/R)^2[\cos^2 \theta_0 \sin^2 \theta + \\
&\sin^2 \theta_0 \sin^2 \phi \sin^2 \theta_0 \cos^2 \theta \cos^2 \phi - \\
&(1/2) \sin 2\theta_0 \sin 2\theta \cos \phi] = 1 \quad . \quad (3)
\end{aligned}$$

This is the surface upon which $g_{00} = 0$ (Fig. 2).

We choose our axes so that the magnetic dipole is along the $\theta = 0$ direction. The dipole magnetic field has polar strength B_0 at $r = R$:

$$A_\theta = 0 \quad (4)$$

$$A_r = 0 \quad (5)$$

$$A_\phi = \frac{B_0 R^3}{2r} \sin^2 \theta \quad , \quad (6)$$

where A_ϕ is covariant in the rotating spherical coordinates. For convenience, we rescale B_0 into dimensionless form as

$$\beta_0 = |q|B_0 R/(2mc^2) \quad . \quad (7)$$

The magnetic moment of this dipole field is

$$|\mathbf{M}| = B_0 R^3/2 \quad . \quad (8)$$

In a high-field neutron star, $\beta_0 \sim 10^{15}$.

The Lagrangian of the charged particle is expressed in terms of the proper time τ , with $x^\mu \equiv dx^\mu/d\tau$. There are four equations of motion including one for each momentum component and one energy-momentum constraint. Since the Lagrangian does not explicitly depend on t , we have

$$\frac{\partial L}{\partial t} = 0 \quad . \quad (9)$$

which implies

$$\frac{dP_0}{d\tau} = 0 \quad . \quad (10)$$

Because

$$d\tau = \frac{dt \sqrt{g_{\mu\nu} (dx^\mu/dt)(dx^\nu/dt)}}{c} \quad , \quad (11)$$

we have also

$$\frac{dP_0}{dt} = 0 \quad , \quad (12)$$

where $P_0 = E$, the energy. The equations of motion for r , ϕ , and θ are non-trivial:

$$\frac{\partial L}{\partial r} - \frac{dP_r}{d\tau} = \frac{\partial L}{\partial \phi} - \frac{dP_\phi}{d\tau} = \frac{\partial L}{\partial \theta} - \frac{dP_\theta}{d\tau} = 0 \quad . \quad (13)$$

The energy-momentum-mass constraint is

$$\begin{aligned}
g^{00}P_0^2 + 2g^{0i}P_0(P_i - \frac{q}{c}A_i) + g^{ij}(P_i - \frac{q}{c}A_i)(P_j - \frac{q}{c}A_j) \\
= (mc)^2 \quad , \quad (14)
\end{aligned}$$

including radial and angular terms, and the mixed rotational $g^{0\theta}$, $g^{0\phi}$ terms. The contravariant metric components are

$$\begin{aligned}
g^{00} &= 1/c^2 \quad , & g^{rr} &= -1/r^2 \quad , \\
g^{\theta\theta} &= -(1 - \omega_\theta^2)/r^2 \quad , & g^{\phi\phi} &= -(1 - \omega_\phi^2)/(r^2 \sin^2 \theta) \quad , \\
g^{0\phi} &= -\omega_\phi/(cr \sin \theta) \quad , & g^{0\theta} &= -\omega_\theta/(cr) \quad . \quad (15)
\end{aligned}$$

Along the actual worldpath in spacetime the energy-momentum constraint is equivalent to $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = c^2$. This condition is valid after varying the action and simplifies the equations of motion.

2.2 Asymptotic orbits

As a check of this dynamical system, we consider briefly the $r \rightarrow \infty$ orbits. The constraint (14) alone is sufficient to give essential information about the trajectory. Note first that $\boldsymbol{\omega}^2 < 1$ for physical trajectories. Since $\boldsymbol{\omega}^2 = \omega_\phi^2 + \omega_\theta^2$, each component must also have magnitude less than one. Then $g^{\theta\theta}$ and $g^{\phi\phi}$ are never exactly zero.

In paper I the natural energy scale for unconfined charged particles at the surface $r = R$ was set by $\epsilon \sim \beta_0$, or $E \sim |q|B_0 R/2$, independent of m (see also Longair [8]). In the limit $r \rightarrow \infty$, Eq. (14) simplifies to

$$P_r^2 = (E/c)^2 - (mc)^2 \quad , \quad (16)$$

where E is naturally $\gtrsim \mathcal{O}(|q|B_0 R)$. In this regime, B_0 is much smaller than at the surface of the star, as r is much larger. The magnetic potential term vanishes as $1/r^2$, while the angular momentum terms vanish as $1/r$. Particles can escape as $r \rightarrow \infty$ if $P_r^2 > 0$ and if they remain within the lightsphere surface $\boldsymbol{\omega}^2 = 1$ (Fig. 2). Both graphically and analytically, it is seen that $(\phi, \theta) \rightarrow (0, \theta_0)$ or $(\pi, \pi - \theta_0)$, as $r \rightarrow \infty$. That is, an escaping particle is forced into the common plane of \mathbf{M} and $\boldsymbol{\Omega}$ and leaves along the rotation axis.

FIG. 2. Lightsphere $r(\theta, \phi)$ defined by Eq.(3), surface of $g_{00} = 0$, for $\bar{\omega} = 0.01$. θ and ϕ in radians.

For sufficiently large r (strictly speaking, for sufficiently large $|\boldsymbol{\omega}|$), the magnetic dipole approximation breaks down, and the field (6) can no longer be used. The field lines are twisted about $\boldsymbol{\Omega}$ by the rotation near the lightsphere, and the particle motion is more complex [9]. Gravitational attraction is negligible compared to the magnetic force for such energetic particles, even far from the sphere.

3. LOCAL SINGLE-BODY STATES

The local Landau states can be obtained by expanding the energy-momentum constraint in a Taylor series around $\mathbf{r} = \bar{\mathbf{r}}$, i.e., let \mathbf{r} go to $\bar{\mathbf{r}} + \Delta\mathbf{r}$. The local point is defined by its spherical coordinates $(\bar{r}, \bar{\theta}, \bar{\phi})$. We retain terms up to second order in $\Delta\mathbf{r}$ and momentum \mathbf{P} , but leave the local metric $g_{\mu\nu}(\bar{\mathbf{r}})$ constant.

3.1 Local coordinates and field

We define a local Cartesian coordinate system so that the z co-ordinate is along the local magnetic field vector, the x coordinate is orthogonal to z and pointing outward, and the y coordinate is azimuthal.

The dipole magnetic field is

$$\mathbf{B} = (B_0/2) \left(\frac{R}{r} \right)^3 [2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}}] \quad (17)$$

In terms of the local displacement $\Delta\mathbf{r}$, the new coordinates are given by

$$\begin{aligned} x &= \frac{\Delta r \sin \theta - 2r \Delta \theta \cos \theta}{\sqrt{1 + 3 \cos^2 \theta}}, \\ z &= \frac{2\Delta r \cos \theta + r \Delta \theta \sin \theta}{\sqrt{1 + 3 \cos^2 \theta}}, \\ y &= -r \sin \theta \Delta \phi, \end{aligned} \quad (18)$$

while the local Cartesian basis is

$$\begin{aligned} \hat{\mathbf{x}} &= \frac{\sin \theta \hat{\mathbf{r}} - 2 \cos \theta \hat{\boldsymbol{\theta}}}{\sqrt{1 + 3 \cos^2 \theta}}, \\ \hat{\mathbf{z}} &= \frac{2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}}}{\sqrt{1 + 3 \cos^2 \theta}}, \\ \hat{\mathbf{y}} &= -\hat{\boldsymbol{\phi}}. \end{aligned} \quad (19)$$

The bar on $\bar{\mathbf{r}}$ is now omitted unless needed.

3.2 Local Landau states

The constraint can be rewritten in the local coordinates as

$$\begin{aligned} &\epsilon^2 - \Pi_x^2 - \Pi_z^2 - (\Pi_y' - \beta x/r)^2 - 1 + \\ &2\omega_\phi \epsilon (\Pi_y' - \beta x/r) - 2\omega_\theta \epsilon \frac{\sin \theta \Pi_z - 2 \cos \theta \Pi_x}{\sqrt{1 + 3 \cos^2 \theta}} + \\ &2\omega_\phi \epsilon \left(\frac{\beta \cos \theta}{r^2 \sin \theta (1 + 3 \cos^2 \theta)^{3/2}} \right) [4 \cos \theta x^2 - \\ &3 \cos \theta \sin^2 \theta z^2 + 2 \sin \theta (3 \cos^2 \theta - 1) xz] + \\ &(\Pi_y' - \beta x/r)^2 \omega_\phi^2 + \\ &\frac{(\sin \theta \Pi_z - 2 \cos \theta \Pi_x)^2}{1 + 3 \cos^2 \theta} \omega_\theta^2 = 0 \quad (20) \end{aligned}$$

Here we use dimensionless local magnetic field strength, energy, and momenta given by

$$\begin{aligned} \beta &\equiv \frac{|q\mathbf{B}|r}{2mc^2} = \beta_0 \left(\frac{R}{r} \right)^2 \sqrt{1 + 3 \cos^2 \theta}, \\ \epsilon &= E/mc^2, \\ \Pi_{x,z} &= P_{x,z}/mc. \end{aligned} \quad (21)$$

Π_y' is the new canonical momentum given by the transformation

$$\Pi_y' = -\frac{P_\phi}{mcr \sin \theta} + \frac{qB_0 R^3 \sin \theta}{2mc^2 r^2} \quad (22)$$

The second term is a shift which is constant in the local coordinate system. The first line of the constraint (20) gives the classical version of the Landau system in terms of the local magnetic field $\mathbf{B}(\bar{\mathbf{r}})$, which also defines the local z axis. The other terms are linear and quadratic corrections (Coriolis and centrifugal terms) due to the local $\boldsymbol{\omega}$ components ω_ϕ and ω_θ .

Neglecting rotational effects yields the classical cyclotron motion [7]. In the semiclassical form, these are the Landau orbitals, with energy eigenvalues

$$\epsilon_0^2 = 1 + \Pi_z^2 + \left(\frac{\hbar}{mc} \right) \left(\frac{2\beta}{r} \right) [2n_L + 1 - \text{sgn}(q)\sigma] \quad (23)$$

where the principal Landau quantum number is $n_L = 0, 1, 2, \dots$ and the spin σ has been included, $\sigma = \pm 1$ [6,10]. For $\text{sgn}(q) < 0$, the state $(n_L, \sigma = 1)$ is degenerate with $(n_L + 1, \sigma = -1)$. The Landau orbitals are characterized by length and momentum scales $L = \sqrt{\hbar c / (2|q\mathbf{B}|)}$ and $P_L = \sqrt{\hbar |q\mathbf{B}| / (2c)}$ in the (x, y) plane transverse to the field. The longitudinal momentum $P_z = mc\Pi_z$ is an eigenvalue continuous over the range $-\infty < P_z < +\infty$, as the motion in that direction is force-free. Note that the magnetic factor proportional to β is suppressed by the ratio of the Compton wavelength $\hbar/(mc)$ to the spherical distance r . The effect of the magnetic field in the energy is thus negligible compared to mc^2 unless β is very large, as seen in paper I. Single-body Landau states of fermions are unstable against radiation, unless the lower energy states are already filled, as they are in Sect. IV below.

We now consider the effect of rotation as a first-order perturbation in $\bar{\omega}$, approaching the problem semiclassically. (The terms quadratic in $\bar{\omega}$ are much smaller.)

The unperturbed orbitals are Landau states in the (x, y) transverse plane. We keep the terms linear in ω (Coriolis effect) and average them over one Landau orbit, as in paper I, Sect. V. But unlike that case, this averaging is over a *local* microscopic orbit, not a macroscopic orbit over the whole sphere. The terms linear in $\bar{\omega}$ thus do not average to zero in general. The modified classical energy-momentum constraint is then

$$\begin{aligned} \epsilon^2 - \Pi_x^2 - \Pi_z^2 - (\Pi_y' - \beta x/r)^2 - 1 - \\ 2\omega_\theta \epsilon \frac{\sin \theta \Pi_z}{\sqrt{1 + 3 \cos^2 \theta}} + \\ 2\omega_\phi \epsilon \left(\frac{\beta \cos \theta}{r^2 \sin \theta (1 + 3 \cos^2 \theta)^{3/2}} \right) \times \\ [4 \cos \theta x^2 - 3 \cos \theta \sin^2 \theta z^2] = 0 \quad , \end{aligned} \quad (24)$$

noting that $\langle x \rangle_0 = \langle y \rangle_0 = \langle \Pi_x \rangle_0 = \langle \Pi_y \rangle_0 = 0$, averaged over an unperturbed cyclotron orbit. These orbits have semiclassical radius $a(n_L) = 2L\sqrt{n_L + 1/2}$. The semiclassical result for the Landau energies corrected through $\mathcal{O}(\bar{\omega})$ is then

$$\begin{aligned} \epsilon_1^2 = \epsilon_0^2 + 2\omega_\theta \epsilon_0 \sin \theta \Pi_z / \sqrt{1 + 3 \cos^2 \theta} - \\ 2\omega_\phi \epsilon_0 \left(\frac{\beta \cos \theta}{r^2 \sin \theta (1 + 3 \cos^2 \theta)^{3/2}} \right) \times \\ [4(2n_L + 1) \cos \theta L^2 - 3 \cos \theta \sin^2 \theta z_C^2] \quad , \end{aligned} \quad (25)$$

where we have used $\langle \Pi_z \rangle_0 = \Pi_z$, $\langle x^2 \rangle_0 = a^2(n_L)/2$, and a distance cutoff z_C for motion in the z direction. (This cutoff is discussed further in Sect. IV below.) The last term can be neglected if the Landau orbits are much smaller than the spherical distance r : $L, z_C \ll r$, as it is another factor of L/r smaller than the zeroth-order term. The energies depend on Π_z^2 at zeroth order but receive a correction proportional to Π_z at first order in $\bar{\omega}$, breaking the symmetry $\Pi_z \rightarrow -\Pi_z$. The Landau pole states of paper I can be recovered if $\sin \theta \rightarrow 0$, $r = R$, identifying n_L with the old n_θ and setting $\Pi_z = 0$.

The error arising from neglecting the cubic terms in the Taylor expansion can be estimated and varies as $\sim |\Delta \mathbf{r}|/r$ times the quadratic terms. That is, the cubic and higher terms in the expansion are suppressed by additional powers of the Landau length L over the sphere size r .

4. LOCAL MANY-BODY STATES

4.1 Density of states

The full density of states is a product of four factors: the density of Landau states, the degeneracy factor \mathcal{D}_\perp in the (x, y) transverse plane for each Landau state, the density of longitudinal states \mathcal{D}_\parallel for z motion, and the spin factor (one for the ground state $n_L = 0$, two otherwise).

The degeneracy of a given Landau state n_L in the transverse plane is

$$\mathcal{D}_\perp = \frac{|q\mathbf{B}|}{2\pi\hbar c} \quad (26)$$

per unit planar area, a result valid in both non-relativistic [11] and relativistic regimes, while the longitudinal motion contributes a factor

$$\mathcal{D}_\parallel = \frac{mc \, d\Pi_z}{2\pi\hbar} \quad (27)$$

per unit longitudinal length. Thus the number of states, including the spin factor, is

$$\frac{d^2 \mathcal{N}}{dS \, dz} = (2 - \delta_{n_L, 0}) \cdot \mathcal{D}_\perp \cdot \mathcal{D}_\parallel \quad , \quad (28)$$

per unit transverse surface area dS and unit longitudinal length dz .

In the semiclassical limit, where n_L is quasi-continuous, the density of Landau states per unit energy is given in dimensionless form by

$$\begin{aligned} \frac{dn_L}{d\epsilon} = \left(\frac{mcr}{4\beta\hbar} \right) [1 + (2\omega_\theta / \sqrt{1 + 3 \cos^2 \theta})]^{-1} \times \\ [1 + \Pi_z^2 + 2\hbar\beta / (mcr) [2n_L + 1 - \text{sgn}(q)\sigma] + \\ 2\omega_\theta \epsilon_0 \sin \theta [\Pi_z / \sqrt{1 + 3 \cos^2 \theta}] \quad . \end{aligned} \quad (29)$$

4.2 Thomas-Fermi approximation

The Thomas-Fermi method approximates quantum many-body fermion states in a varying potential with local states defined by a locally constant field [6]. In our case, the local electron states are filled up to some highest and partially-filled Landau level n_L^* by the charge carriers, assumed here to be electrons. That part ζ^2 of the squared energy ϵ^2 arising from the Π_z terms alone,

$$\zeta^2(\Pi_z) = \Pi_z^2 + 2\omega_\theta \epsilon \sin \theta \Pi_z / \sqrt{1 + 3 \cos^2 \theta} \quad ,$$

must be cut off at some maximum, as must the corresponding z motion. In a real system, the cutoffs are provided naturally by the presence of lattice ions: $|z_C| \gtrsim \hbar / (Z_{\text{eff}} \alpha m c)$ (Bohr length) and $|\Pi_z| \gtrsim Z_{\text{eff}} \alpha$, where Z_{eff} is an effective (screened) positive ionic charge [6]. In the realistic case the z_C term is unnecessary; assume that $\zeta^2 < \zeta_F^2 = (Z_{\text{eff}} \alpha)^2$. Then the longitudinal momentum Π_z is cut off asymmetrically at $\Pi_z = \Pi_z^+ > 0$ and $\Pi_z^- < 0$, with $\Pi_z^+ \neq -\Pi_z^-$ if $\bar{\omega} \neq 0$ (Fig. 3).

FIG. 3. Momentum-dependent part ζ^2 of the squared energy ϵ^2 as a function of longitudinal momentum Π_z , including $\mathcal{O}(\bar{\omega})$ correction, showing the solutions Π_z^\pm of $\zeta^2 = \zeta_F^2$.

The electron number density n_e at any point is given by

$$n_e = \frac{|q\mathbf{B}|}{2\pi\hbar c} [1 + 2(n_L^* - 1) + 2\nu] \cdot \Delta\Pi_z (mc/2\pi\hbar) \quad , \quad (30)$$

where ν is the partial filling factor of the highest Landau level n_L^* , $0 \leq \nu \leq 1$, and n_L^* is assumed ≥ 1 . (If $n_L^* = 0$, the entire term in brackets is replaced simply by ν .) Because of the z degree of freedom, each Landau level is actually a band, with lowest energy level at some $\Pi_z \neq 0$, whose sign depends on that of ω_θ . In addition, there is the planar degeneracy, modified by the partial filling factor ν in the highest Landau level. If n_e is specified along with $\Delta\Pi_z$, then n_L^* , ν , and ϵ_F can be determined. Procedural details are found in the Appendix.

At zero temperature (the case examined in this paper), the electrons cannot radiate into already-filled lower-energy single-body states. Radiation occurs only if fermions are externally excited above the Fermi energy.

If n_e is held constant over a sphere of radius r , then n_L^* , ν , and ϵ_F change over that surface as $|\mathbf{B}|$ changes. The variation of n_L^* and ν over the surface is given by

$$\begin{aligned} n_L^* &= \text{Int} \left[\left(\frac{2\pi\hbar}{mc} \right)^3 \left(\frac{n_e}{\Delta\Pi_z} \right) \left(\frac{B_c}{B_0\sqrt{\alpha}} \right) \left(\frac{r}{R} \right)^3 \times \right. \\ &\quad \left. \frac{1}{4\sqrt{1+3\cos^2\theta}} + \frac{1}{2} \right] \quad , \\ \nu &= \left(\frac{2\pi\hbar}{mc} \right)^3 \left(\frac{n_e}{\Delta\Pi_z} \right) \left(\frac{B_c}{B_0\sqrt{\alpha}} \right) \times \\ &\quad \frac{1}{4\sqrt{1+3\cos^2\theta}} + \frac{1}{2} - n_L^* \quad , \end{aligned} \quad (31)$$

where $\text{Int}[f]$ in (31) denotes the integer value of the function f . Typical variations of n_L^* and ν over the sphere are shown in Figs. 4 and 5. The Fermi energy at any point on the surface is

$$\epsilon_F^2 = 1 + \zeta_F^2 + \left(\frac{B_0\sqrt{\alpha}}{B_c\pi} \right) \left(\frac{R}{r} \right)^3 (2n_L^* + 1 + \sigma) \times \sqrt{1+3\cos^2\theta} \quad , \quad (32)$$

where $\text{sgn}(q) = -1$, and $\sigma_F = -1$ for the n_L^* level; the critical field strength is $B_c = (m_e^2/\pi)\sqrt{c^5/\hbar^3} = 1.3 \times 10^{12}$ G, defined in paper I. Matching variations of ϵ_F are shown in Fig. 6. As the Fermi energy ϵ_F varies over a sphere, electron currents flow as the electrons seek the lowest energy. There are also radial currents (see Sect. IV.C below).

Quantitative details of the subsequent evolution [12] are beyond the scope of this paper, but certain qualitative features are clear. The magnitude and evolution of the currents depend on the electrical conductivity, which

in turn depends on the nuclear crystal and the magnetic field. (Relativistic electrons travel at essentially the speed of light, leading to saturated current densities of $\mathcal{O}(n_e c)$, apart from magnetic effects on the density of states [1]). As the magnetic field and rotation affect the electron Fermi energy, the beta equilibria of neutrons [13] and muons are also affected above their respective thresholds. This last effect cannot be included without revising the hadronic equation of state [2]. The Fermi energy will be unequal over the sphere at first, a limit expected to apply to young neutron star crusts. With time, ϵ_F will equilibrate to the same value everywhere, and the \mathbf{B} and n_e , n_p profiles will change in parallel.

FIG. 4. The last, partially-filled Landau level n_L^* on a spherical surface, at constant density, defined by Eq. (31); $B_0 = 10B_c$. (a) $\rho = 10$ g/cm³; (b) $\rho = 10^7$ g/cm³; (c) $\rho = 10^{13}$ g/cm³. θ and ϕ in radians.

FIG. 5. The partial filling factor ν of the last Landau level n_L^* on a spherical surface at constant density, defined by Eq. (31); $B_0 = 10B_c$. (a) $\rho = 10 \text{ g/cm}^3$; (b) $\rho = 10^7 \text{ g/cm}^3$; (c) $\rho = 10^{13} \text{ g/cm}^3$. θ and ϕ in radians.

FIG. 7. The electron number density n_e on spherical surface at constant Fermi energy, defined by Eq. (30); $B_0 = B_c$. Units are inverse Compton volume. (a) $\epsilon_F = 1.06$; (b) $\epsilon_F = 5.0$; (c) $\epsilon_F = 1800$. θ and ϕ in radians.

4.3 Radial structure

For a simple neutron star, we can assume a given profile of positive ions, and thus electrons, with radius r . With the field $|\mathbf{B}(\mathbf{r})|$ in addition, the radial dependence of ϵ_F can be found.

In Fig. 8 the radial profile is shown for ϵ_F from $r/R = 0.7$ to 1.0 , with an expanded subfigure of $r/R = 0.9998$ – 1.0 . Even at this detail, the thin surface non-relativistic layer cannot be seen, but the quantum regime of discretized ϵ_F steps is clearly visible. Fig. 9 shows ϵ_F as a function of r/R and θ . The rotational correction drops out in certain cases. If $\sin \phi = 0$ ($\bar{\mathbf{r}}$ in the \mathbf{M} – $\mathbf{\Omega}$ plane) or $\sin \theta_0 = 0$ (no tilt), ω_θ vanishes. Also, if $\sin \theta = 0$ ($\bar{\mathbf{r}}$ along the dipole axis), $P_\theta \sim r \sin \theta \cdot P_z \rightarrow 0$, and the Coriolis effect disappears. Otherwise the ϵ_F radial profile is affected dramatically by the rotational correction for relativistic ϵ_F .

FIG. 6. The dimensionless Fermi energy ϵ_F on spherical a surface at constant density, defined by Eq. (32); $B_0 = 10B_c$. (a) $\rho = 10 \text{ g/cm}^3$; (b) $\rho = 10^7 \text{ g/cm}^3$; (c) $\rho = 10^{13} \text{ g/cm}^3$. θ and ϕ in radians.

The opposite limit is the case where the Fermi energy is constant in space and no currents flow, a situation probably holding for crusts at later times. Since the field $|\mathbf{B}|$ varies, the electron density n_e , and thus the positive ion density n_p , must also vary. This is the case where the magnetic field is strong enough to dominate the mechanical structure of dense matter. We show three examples of how n_e is redistributed over a sphere of constant r for fixed ϵ_F (Fig. 7). The proton density is $n_p = (Z/Z_{\text{eff}})n_e$, and ρ can be inferred from Eq. (A2).

FIG. 8. Dimensionless Fermi energy ϵ_F as function of r/R for simplified neutron star crust model of text; $\sin \theta = \sin \phi = 1$, and $B_0 = 10B_c$. (a) $r/R = 0.7$ – 1.00 ; (b) $r/R = 0.9998$ – 1.00 . Terraced steps indicate quantum regime at the surface.

FIG. 9. Dimensionless Fermi energy ϵ_F as a function of r/R and θ for simplified neutron star crust model of text. (a) $\sin \phi = 1$, $r/R = 0.7-1.00$, $B_0 = 10B_c$, $\sin \theta > 0$; (b) detail showing $\sin \theta > 0.2$. θ and ϕ in radians.

For this crust profile, we have used a neutron star model of Glendenning [3], with only electrons, nuclei, and neutrons, and no muons or other hadrons. The radius and mass of the neutron star are $R = 11$ km and $M = 1.55M_\odot$, respectively. The profile starts at $r/R = 0.7$, where $\rho = 2 \times 10^{13}$ g/cm³ and decreases outward. As ϵ_F varies in space, currents flow in response, a scenario probably relevant to early neutron star crusts. (Recall that we ignore the effect of gravity.) Again, the neutron and muon beta equilibria are changed from their zero-field analogues.

Now consider the opposite limit again, probably holding for older crusts, a constant ϵ_F , with $n_e(r)$ determined by a given field profile $\mathbf{B}(\mathbf{r})$. Then ν must be set independently; here we take $\nu = 0.1$ and 0.9 as illustrative. For critical field strength, the density *rises* with r , while for even higher fields, the density is quenched to essentially a constant (Fig. 10). For the same surface value of ϵ_F , the n_e density of Glendenning [3] is too large to be shown on the scale of Fig. 10. Thus, the crust densities are low compared to the non-magnetized values when the magnetic field is the controlling factor.

FIG. 10. The electron number density n_e as a function of r/R for constant $\epsilon_F = 1.06$; $\sin \theta = \sin \phi = 1$, and $B_0 = 10B_c$. Solid: $\nu = 0.1$; dotted: $\nu = 0.9$. Density profile of simplified neutron star crust model in text is too large to show on this scale. Units are inverse Compton volume.

5. CONCLUSION

This concludes the treatment of semiclassical orbitals begun in paper I. In this paper, we have found the local one- and many-body states of relativistic charged particles confined to a sphere with an intense, rotating magnetic dipole field.

There remain full quantization with the Dirac equation and the inclusion of the positively-charged lattice structure to determine the local neutron star matter state. A full calculation requires a self-consistent treatment of gravity, nuclear matter, magnetic fields, and currents,

with chemical equilibrium and Coulomb neutrality. Depending on the degree of lattice disorder and interelectron forces, various conducting, insulating, or quantum Hall-like many-body states can arise [11,14], affecting the macroscopic currents inferred in Sect. IV.

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APPENDIX: MANY-BODY THEORY

To determine the filling of momentum states labelled by Π_z , the cutoff equation $\zeta^2(\Pi_z) = \zeta_F^2$ must be solved for some given value of ζ_F^2 , taken here as $(Z_{\text{eff}}\alpha)^2$. At zeroth order in $\bar{\omega}$, this equation is trivial. Once the rotational corrections are included, the equation is not only quadratic and linear in Π_z , but has an implicit dependence on Π_z through ϵ_0 .

In the $\mathcal{O}(\bar{\omega})$ correction, our procedure is to take $\epsilon_0^2 = 1 + \epsilon_L^2$, where ϵ_L^2 is the purely two-dimensional Landau term:

$$\epsilon_L^2 = \frac{\hbar|q\mathbf{B}|}{m^2c^3}[2n_L + 1 - \text{sgn}(q)\sigma]$$

and neglect the Π_z^2 term, as the latter is typically much smaller than one. In that case, the cutoff equation $\zeta^2(\Pi_z) = \zeta_F^2$ is a simple quadratic with the two roots Π_z^\pm and

$$\Delta\Pi_z = \Pi_z^+ - \Pi_z^- = 2\sqrt{\zeta_F^2 + \omega_\theta^2(1 + \epsilon_L^2)\sin^2\theta/(1 + 3\cos^2\theta)} \quad (33)$$

as the allowed spread of z momenta. Given a value of n_e , the values of ϵ_F , n_L^* , and ν are found iteratively, starting with $\bar{\omega} = 0$, then with this solution used in the $\mathcal{O}(\bar{\omega})$ corrections.

In the opposite case, of fixed ϵ_F , n_L^* is determined, while we set $\nu = 0.1$ and 0.9 as illustrative (Fig. 10). The density n_e is then found. As long as $\nu \neq 0.5$, n_e can fall or rise with r . A complete treatment of the bulk crust requires inclusion of nuclear matter and gravity [2], as well as an interior \mathbf{B} field profile, not necessarily a dipole.

A semi-realistic spatial profile of electron density requires the proton density $n_p = n_e$, usually determined in terms of mass density ρ . The “effective” electron density n_e , the density available for conduction, is

$$(2\pi\hbar/mc)^3 n_{e,\text{cond}} = \frac{Z_{\text{eff}}}{A} \cdot \frac{1.1 \times 10^{-5} \rho}{\text{g cm}^{-3}}, \quad (34)$$

where Z_{eff} is the number of electrons per nucleus available for conduction (unbound electrons). The atomic number $A = N + Z$, where the neutron number N per nucleus is abnormally large for nuclei in an electron Fermi sea. The number density is normalized to a Compton volume $(2\pi\hbar/mc)^3$. From $\rho = 10$ to about 3×10^4 g/cm³, the inner electrons of the atoms remain bound, not participating in conduction; in this case, $Z_{\text{eff}} < Z$ and can be read off from standard atomic structure [15]. The nuclei are always iron ($Z = 26$ and $Z_{\text{eff}} = 8, 16, 24$) at these densities [3,18]. For higher densities, the orbitals of different atoms merge, and $Z_{\text{eff}} = Z$. (We neglect the formation of partially ionized atoms at ultrahigh densities, assuming that all electrons are stripped from their nuclei.) The results do not depend sensitively on Z_{eff} .

For nuclear matter composition at densities below complete nuclear dissociation, but for $Z > 26$, we use the results from [16,17] (see also [18]) at $\rho < 2 \times 10^{13}$ g/cm³. Our method can be applied at higher densities with free nucleons, but it needs to incorporate the presence of muons and then of heavier strange and non-strange hadrons, and then possibly of a quark-gluon plasma [3]. In this paper, only the simplest case of electrons and positive ions is examined.

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